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Rapid relaxation in a one-dimensional gravitating system

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The relaxation of one-dimensional gravitating systems has been studied over the past three decades. The numerical efficiency with which these models can be simulated makes them ideal for studying long time evolution of gravitational systems. Much controversy has surrounded the relaxation time for the one-dimensional system of N parallel mass sheets. Early work suggested a relaxation on the order of N^2 characteristic times; however, subsequent simulations did not bear this out. Instead, it has been shown that relaxation, if it occurs at all, takes on the order of 10^7 characteristic times. Here we consider the relaxation of a different one-dimensional system consisting of concentric spherical mass shells. Past studies have shown the two shell system has more robust ergodic properties than its planar counterpart, possibly suggesting a more rapid relaxation for the N shell system. We found that the simulation for 4, 16, and 64 shells converges to the density obtained in the Vlasov limit on a much shorter time scale with an upper bound of approximately 10^5 characteristic times. This is orders of magnitude smaller than in the sheet system. [S1063-651X(97)51011-7]

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I. INTRODUCTION

An important question in gravitational dynamics concerns the amount of time that a system takes to reach equilibrium. One-dimensional models, due to their numerical efficiency, are ideal for studying the long time evolution for gravitational systems. The most popular model has been the one-dimensional system consisting of N parallel mass sheets [1–5]. It has been suggested that the sheet system could serve as a model to test astrophysical theories concerning the approach to equilibrium for stellar systems; however, simulations show that if the system reaches equilibrium at all, it takes a very long time. This casts into question its usefulness as a test model and has stimulated a search for other systems which unambiguously relax to equilibrium.

Reidl and Miller [6] have compared the Vlasov density distribution for a one-dimensional system of parallel mass sheets to the distribution obtained from a dynamical computer simulation of the system with 100 sheets. In most cases, depending on the initial conditions, they found a significant difference between the equilibrium Vlasov distribution and the average distribution of particles determined from dynamical simulations lasting on the order of thousands of crossing times. Accompanying this failure to relax in the predicted time scale, they found correlations in position and velocity that persisted for the duration of the runs. A later study of the same system suggests that the time scale for relaxation, if it exists, is on the order of 10^7 characteristic times [7]. It has been conjectured that preferred, ''sticky'' regions in the phase space of this system may be responsible for the long relaxation time scale [8,9]. In a system consisting of three mass sheets Froeschle and Scheidecker found that only 4% of the phase space was actually occupied by chaotic orbits [3].

Recently, we studied a system of two concentric spherical mass shells and found a much larger chaotic component in the phase space [10]. This suggests that the spherical shell system may reach equilibrium more quickly, if the correlations between particles (shells) do not persist for long times, as they do in the sheet system. In this communication we report the results of our studies of larger shell systems. We determine the Vlasov density distribution for the spherical shell system and compare it to the average particle distribution from simulations of systems with 4, 16, and 64 shells. For the case of the planar system Rybicki derived exact microcanonical distributions for the N particle system and showed that the Vlasov limit was approached as $N\rightarrow\infty$ [11]. We find that, as the number of shells in the system increases, the simulation distribution approaches the Vlasov distribution and, even for only 64 particles, the agreement is quite good. Moreover, in each instance of 4, 16, and 64 particles, the final distribution from the simulation remains the same when the initial conditions are drastically changed. For all simulations, the time required for the shell system to converge to the final distribution is significantly smaller than in the parallel sheet system. This robust convergence suggests that the spherical shell system may be a more useful model for testing astrophysical theories concerning the evolution of stellar systems.

II. DESCRIPTION OF SPHERICAL SHELL MODEL

We consider a system of concentric spherical mass shells of equal mass and uniform mass density. The motion of each shell is purely radial; there is no rotational motion. The shells move under their mutual and self gravitation, with the acceleration of the ith shell given by

$$a_i = -Gm(i-\tfrac{1}{2})/r_i^2, \tag{1}$$

where the shells are numbered $i=1,2,...,n$ with $i=1$ corresponding to the innermost shell. G is the universal gravitational constant, m is the mass of an individual shell, and r_i is the radius of the ith shell. In this study, we chose units such that $G=1$, the total mass is 1, and the individual mass m is $1/n$. The energy of the system is conserved and is given by

$$E=\sum_{i=1}^{n} \tfrac{1}{2}mv_i^2-Gm^2(i-\tfrac{1}{2})/r_i\,. \tag{2}$$

The system is contained in a spherical box with reflecting boundaries. In addition, there is a reflecting inner boundary around the origin of the system to avoid numerical difficulties that arise from the singularity of the gravitational force.

As the shells move, they can intersect and pass through one another. At the point of crossing, there is a discontinuity in the amount of mass contained within a given shell's radius. For example, at a crossing the $i=2$ shell becomes the $i=1$ shell and the interior mass associated with that shell decreases from m to zero. This discontinuity in interior mass at a crossing gives rise to a discontinuity in each shell's acceleration. Between crossings, the acceleration of a shell is continuous and its energy is conserved.

We have developed an algorithm that allows for the evolution of the system without numerically integrating the equations of motion. From the conservation of energy of each shell between crossings, it is possible to solve for the time that the ith shell will reach a specified radius r_i, given the initial time $t_{0,i}$, radius $r_{0,i}$, and shell energy E_i, by performing the integration

$$t_t-t_{0,i}=\pm\int_{r_{0,i}}^{r_i} dr\left[(2/m)\left(E_i+\frac{\alpha_i}{r}\right)\right]^{-1/2} \tag{3}$$

where $\alpha_i=Gm^2(i-\frac{1}{2})$. The proper sign is determined by previous knowledge of whether the shell is moving inward $(-)$ or outward $(+)$. Two possible outcomes of the integration in the above equation are possible, depending on the sign of the energy of the shell.

The event-driven algorithm used in this study is based on the ability to calculate the time a shell reaches a specific radius. There are three possible events for each shell: collision with the boundary, turning point (only possible for a shell with negative energy and positive velocity), and crossing with another shell. The basic strategy for the algorithm is to determine the times at which these events occur for all shells and then sort these times to find the next event. The appropriate initial conditions are updated and the process is repeated. This algorithm for following the evolution of the system has many advantages over one that numerically integrates the equations of motion. First, the efficiency of solving only for events allows the system to be followed for a much longer time. In addition, the truncation error associated with any numerical integration scheme is avoided. Computer round-off error is also reduced since it is introduced only at events instead of at every time increment, as is the case with numerical integration methods.

III. COMPARISON OF VLASOV AND SIMULATION DENSITY DISTRIBUTIONS

To predict a density profile for the shell system, we considered the system in the Vlasov limit of an infinite number of particles with the mass of the individual particles approaching zero. In this approximation, the motion of the system is described by a simple fluid flow. The Vlasov one-dimensional density $\rho(r)$ for the spherical shell system is given by the second-order differential equation,

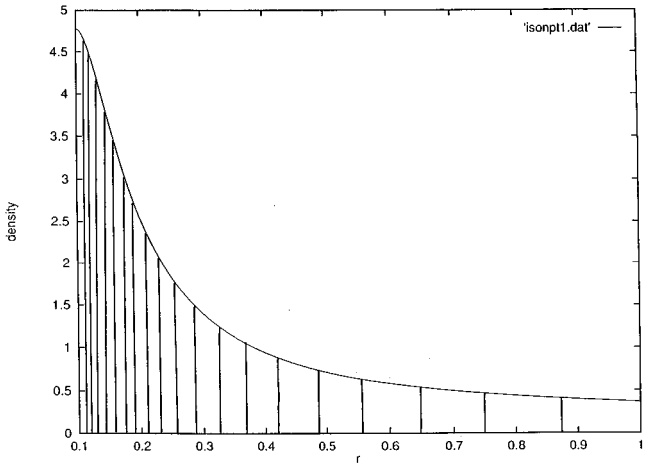

FIG. 1. The density distribution predicted from the Vlasov theory for the spherical shell system (total energy $=-0.8$, inner boundary radius $=0.1$) is shown. The vertical lines show the boundaries of the equal probability cells used to determine the average density distribution for the simulation. All units are dimensionless.

$$\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{d\rho}{dr}\right]=-G\beta\rho(r), \tag{4}$$

where $\beta=m/k_BT$ and k_B is Boltzmann's constant [12]. Note that this is different than the governing equation for the three-dimensional system of isothermal spheres that has been explored in the literature [12]. Equation (4) can be reduced to a set of two coupled first-order differential equations in the variable $g=\beta\rho$ and solved by numerical integration (we chose Burlisch-Stoer [13] for its accuracy and efficiency) with the appropriate boundary conditions at the inner boundary radius a. The derivative dg/dr at $r=a$ is zero, and $g(a)$ is chosen to yield the desired total energy of the system. The energy of the Vlasov distribution is given by

$$E=[b\rho(b)-a\rho(a)]/\beta-1/2\beta, \tag{5}$$

where a is the radius of the inner boundary and b is the radius of the outer boundary. The total mass of the system M, is set to 1 and we determine β from integrating $g(r)$,

$$M=\int_a^b \rho(r)dr=\beta\int_a^b g(r)dr=1. \tag{6}$$

Once $g(r)$ and β are determined, $\rho(r)$ can be calculated and the total energy of the system can be determined. Figure 1 shows $\rho(r)$ for an inner boundary radius of 0.1 and a total energy of -0.8.

To determine how well this compares to a dynamical simulation with a finite number of particles, we divided the space between the inner and outer boundaries into 20 cells of equal probability, i.e., the area under the Vlasov density curve is the same for each cell (see Fig. 1). The initial positions and velocities for the simulation were assigned according to the total energy and inner boundary radius specified for the Vlasov distribution. The shells' positions were equally spaced between the inner and outer boundary and the difference between the total system energy and their total potential energy was divided equally by assigning equal ve-

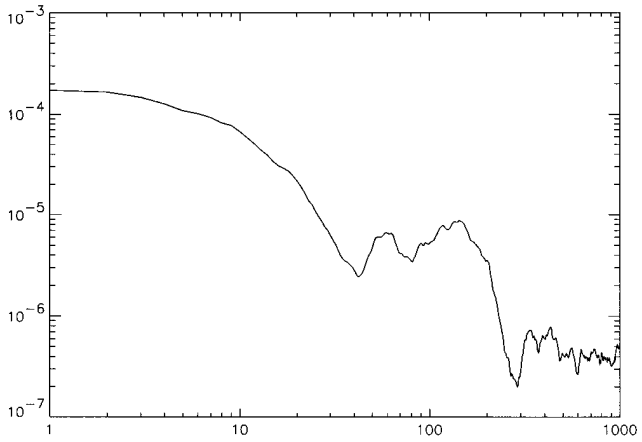

FIG. 2. The variance between simulation (64 particles, total energy=0.8, inner barrier radius=0.1) and Vlasov prediction is plotted vs time. All units are dimensionless. Note that in 800 time units the fluctuations in variance have died out and the system has relaxed.

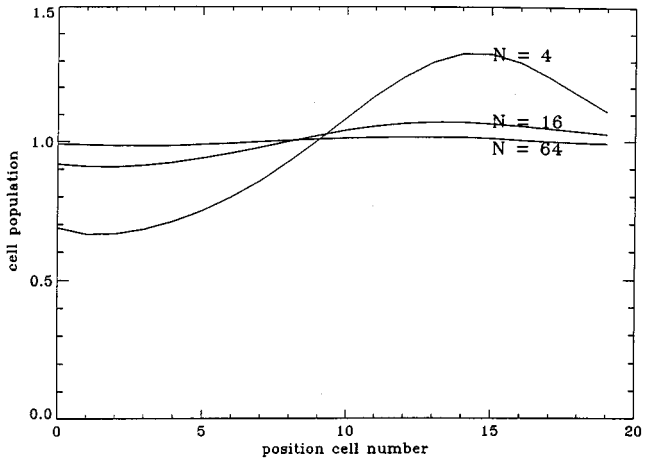

FIG. 3. The average density distribution for the simulation of 4 particles, 16 particles, and 64 particles is shown. All units are dimensionless. The dashed line shows the density distribution obtained from the Vlasov theory. Convergence to the Vlasov limit is observed for increasing population.

locities to all shells. The system was sampled at equal time intervals (every 0.1 time unit in our system of units) and the number of particles was accumulated in the appropriate cell according to the position of the shell. To determine when to stop the simulation, a comparison was made between the current cell population and the cell population at half the current number of samples. All cell populations were normalized by dividing the cell population by the total number of samples. During a sample, $P_i = P_i(t)$ is the current (normalized) population of cell i, and P_i' is the (normalized) population of cell i at half of the current number of samples, i.e., $P_i' = P_i(t/2)$. The simulation was stopped when the sum of the square of deviations between the current population and the halfway point population,

$$\sigma^2 = \sum_{i=1}^{20} (P_i - P_i')^2/20, \tag{7}$$

fell below 1.0×10^{-8}. In Fig. 2 we replace P_i' with 1, the predicted Vlasov result, and plot the variance as a function of time for the 64-particle system. It is apparent from the figure that the fluctuations gradually vanish, indicating the approach to equilibrium, but the decay does not follow a simple functional form. Therefore any definition of the relaxation time based on this function is arbitrary. The criteria we chose for terminating the simulations guarantees that we are well past the point where the system has settled down and thus provides an upper bound for any reasonable choice of relaxation time.

Figure 3 shows the simulation results for systems with 4, 16, and 64 shells. The horizontal line indicating a population of one for all cells represents the Vlasov density distribution. The distribution for the four-particle system shows a marked population deficiency in the inner cells and a surplus in the outer cells compared with the Vlasov density. The 16-particle case also shows a similar structure, but the difference between simulation and Vlasov prediction is relatively smaller. For 64 particles, the density distribution shows excellent agreement with the Vlasov prediction. This agreement is surprising since the Vlasov theory assumes that the number of particles is infinite.

It is important to note that, in general, this agreement was not found in the study of the sheet system by Reidl and Miller [6]. They studied systems with 100 sheets with five different initial conditions and each case demonstrated a different density distribution at the end of the simulation. In four out of the five different initial conditions, their simulations did not agree with the Vlasov prediction. These simulations were carried out for about 1000 characteristic time units (one characteristic time for the sheet system is the time that a sheet takes to traverse the system). The differences in density distributions for different initial conditions suggest that the system still ''remembers'' the initial condition and has not yet reached equilibrium. To test the spherical shell model, we created different initial conditions for the same energy and inner boundary radius and ran the simulation again. The initial virial ratio varied from 0.06 to 1.3 for the two different initial conditions. All three cases (4, 16, and 64 particles) were run with the new initial conditions and the density distributions were found to be almost identical to the ones shown in Fig. 3.

The relaxation time scales for the parallel sheet and spherical shell systems are significantly different. It has been shown by simulation that the sheet system may reach equilibrium on a time scale of 1×10^7 characteristic times. A commonly used characteristic time for spherical systems is the amount of time that a cold system (no kinetic energy) takes to collapse and then expand back to its original distribution. We found that in all instances the upper bound criteria ($\sigma^2 < 1.0\times10^{-8}$) was attained in less than 4×10^5 characteristic times, about two orders of magnitude less than that required for relaxation of the sheet system. The duration of the simulations carried out by Reidl and Miller, in which relaxation was not observed, was on the order of 1000 units. In contrast, it can be seen from Fig. 2 that relaxation of the shell system was completed in about 800 units.

IV. CONCLUDING REMARKS

The evolution of the spherical shell system differs from the parallel sheet system in two respects. First, the relaxation time is much less than two orders of magnitude smaller for the shell system; second, for the energy considered, memory of the initial state is lost much more rapidly for the shell system.

Several factors could be responsible for the differences in behavior between the parallel sheet and spherical shell system. First, the parallel sheet system does not have the reflective boundaries that are present in the shell system. These boundaries induce discontinuities in the shells' velocity, which could aid the system in reaching equilibrium more quickly. The boundaries may also hinder the development of a core-halo structure that develops in the sheet system. It has been suggested that the weak interaction between core and halo is responsible for the slow relaxation time of the sheet system [8]. However, core-halo structures also develop in the one-dimensional spherical shell systems studied by Henon [14] and Yangurazova and Bisnovatyi-Kogan [15] and yet both studies find evidence of a stationary state. These models do not have an outer boundary and shells may escape from the system so their equilibrium is not defined. The spherical models used in these studies differ from ours by including an angular momentum term in the equations of motion which effectively shields the singularity at the center of the system. Recently these models possessing angular momentum have been used to investigate the self-similar structure [16] and Lynden-Bell theory of violent relaxation [17] of spherical mass distributions.

In the spherical shell model the Vlasov approximation for the density profile is approached under the specified conditions for total energy and inner boundary radius as the number of particles increases from 4 to 64. Similar behavior was recently observed in the planar sheet system [18]. We note that the average density profiles obtained from the simulation do not change when new initial conditions (that still retain the total energy and inner boundary radius) are imposed. This convergence to the same density profile for different initial conditions gives added confidence that the system is ergodic and sampling the complete phase space. In later work we will show that a phase transition is predicted in the Vlasov limit for the shell system for certain values of energy and inner boundary radius. In this paper, however, we have chosen system parameters far from this transition in order to study the system isolated from its influence.

An open question for both the planar sheet and the spherical shell systems concerns the population dependence of the system relaxation time. In each case, randomness is introduced into the dynamics by the crossing of a pair of particles (sheets or shells). Whenever a crossing occurs, there is a discontinuous change in the acceleration of each particle. Recently, for the sheet system, one of us has shown that if the crossing times are random then, in the large N limit, the acceleration of a particle becomes a diffusion [19]. The consequence is a relaxation time which scales with N. Since the discontinuity is common to both systems, the arguments used apply equally to the system considered here. The validity of this theory has yet to be tested and is the subject of future work.

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